κ-deformed oscillators, the choice of star product and free κ-deformed quantum fields

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Abstract

The aim of this paper is to study in $D = 4$ the general framework providing various κ-deformations of field oscillators and consider the commutator function of the corresponding κ-deformed free fields. In order to obtain free κ-deformed quantum fields (with c-number commutators) we proposed earlier a particular model of a κ-deformed oscillator algebra (Daszkiewicz M, Lukierski J and Woronowicz M 2008 *Mod. Phys. Lett.* A **23** 9 (arXiv:hep-th/0703200)) and the modification of κ-star product (Daszkiewicz M, Lukierski J, Woronowicz M 2008 *Phys. Rev.* D **77** 105007 (arXiv:0708.1561 [hep-th])), implementing in the product of two quantum fields the change of standard κ-deformed mass-shell conditions. We recall here that other different models of κ-deformed oscillators recently introduced in Arzano M and Marciano A (2007 *Phys. Rev.* D **76** 125005 (arXiv:0707.1329 [hep-th])), Young C A S and Zegers R (2008 *Nucl. Phys.* B **797** 537 (arXiv: 0711.2206 [hep-th])), Young C A S and Zegers R (2008 arXiv: 0803.2659 [hep-th]) are defined on a standard κ-deformed mass shell. In this paper, we consider the most general κ-deformed field oscillators, parametrized by a set of arbitrary functions in 3-momentum space. First, we study the fields with the κ-deformed oscillators defined on the standard κ-deformed mass shell, and argue that for any such choice of a κ-deformed field oscillators algebra we do not obtain the free quantum κ-deformed fields with the c-number commutators. Further, we study κ-deformed quantum fields with the modified κ-star product and derive a large class of κ-oscillators defined on a suitably modified κ-deformed mass shell. We obtain a large class of κ-deformed statistics depending on six arbitrary functions which all provide the c-number field commutator functions. This general class of κ-oscillators can be described by the composition of suitably defined κ-multiplications and the κ-deformation of the flip operator.

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1. Introduction

The standard relativistic local quantum fields on Minkowski space provide a basic tool for the description of fundamental interactions. If we take into consideration the quantum gravity effects such as classical description break down at Planck distances ($\approx 10^{-33}$ cm), it is quite plausible that the space time becomes noncomutative (see e.g. [6, 7]). The κ-deformation of the Minkowski space [8]–[10] and the corresponding quantum κ-deformation of relativistic symmetries (see e.g. [9, 11, 12]) provide a fundamental mass scale and possible tools for the description of the Planckian quantum-gravitational regime.

At present an important task is the construction of κ-deformed quantum field theory on κ-deformed Minkowski space. In contrast with a simpler, recently studied case of θ-deformed symmetries (with constant commutator $[\hat{x}_\mu, \hat{x}_\nu] = \theta_{\mu\nu}$; see e.g. [13, 14]), the κ-deformation of relativistic symmetries is not described by a twist factor[1], and for the κ-Poincaré algebra the universal R-matrix is not known. We can therefore apply mainly the technique of star product (for application to κ-deformation see e.g. [17]–[22]) as representing the κ-deformation of space time.

1.1. Summary of the previous results [1, 2, 25]

The main aim of our scheme presented earlier in [1, 2, 25] was the construction of free quantum κ-deformed fields characterized by the c-number commutator function[2]. It should be stressed that for such free κ-deformed quantum fields all other properties known from the standard free quantum field theory as the notion of locality, microcausality, the structure of Fock space, or kinematic independence of field excitations defining the multi-particle states are modified. The c-number commutator of quantum κ-deformed fields is obtained by the interplay of the two following sources of noncommutativity: the quantum nature of space time and the specific κ-deformation of the field oscillators algebra.

1.1.1. Noncommutativity of space time.　We replace the standard quantum field arguments x_μ by κ-Minkowski noncommutative coordinates $\hat{x}_\mu$

$$[\hat{x}_0, \hat{x}_i] = \frac{\mathrm{i}}{\kappa}\hat{x}_i, \qquad [\hat{x}_i, \hat{x}_j] = 0. \tag{1}$$

One can introduce the Fourier expansion of the κ-deformed free quantum fields

$$\hat{\varphi}(\hat{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^4 p\, \hat{A}(p_0, \vec{p})\, \delta\left(C_\kappa^2(\vec{p}, p_0) - M^2\right) \vdots \mathrm{e}^{\mathrm{i} p_\mu \hat{x}^\mu} \vdots, \tag{2}$$

where the symmetrized κ-deformed plane wave looks as follows (see e.g. [26]):

$$\vdots \mathrm{e}^{\mathrm{i} p_\mu \hat{x}^\mu} \vdots = \mathrm{e}^{\frac{\mathrm{i}}{2} p_0 \hat{x}_0} \mathrm{e}^{\mathrm{i} p_i \hat{x}_i} \mathrm{e}^{\frac{\mathrm{i}}{2} p_0 \hat{x}_0}. \tag{3}$$

The operators $\hat{A}(p_0, \vec{p})$ describe the quantized field oscillators and $C_2^\kappa(\vec{p}, p_0)$ represents the κ-deformed mass square Casimir

$$C_2^\kappa(\vec{p}, p_0) = \left(2\kappa \sinh\left(\frac{p_0}{2\kappa}\right)\right)^2 - \vec{p}^2, \tag{4}$$

defining the κ-deformed mass-shell condition

$$C_2^\kappa(\vec{p}, p_0) - M^2 = 0, \tag{5}$$

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[1]　We restrict ourselves here to twists $T \in \mathcal{U}(P_4) \otimes \mathcal{U}(P_4)$ where $\mathcal{U}(P_4)$ describes the enveloping $D = 4$ Poincaré algebra. For twists not satisfying this condition see [15, 16].
[2]　We extend to noncommutative quantum fields the known definition of (generalized) free fields as described by a c-number commutator (see e.g. [23, 24]).

which implies the energy–momentum dispersion relation

$$p_0 = \pm\omega_\kappa(\vec{p}), \tag{6}$$

where

$$\omega_\kappa(\vec{p}) = 2\kappa\,\text{arcsinh}\left(\frac{\omega(\vec{p})}{2\kappa}\right); \qquad \omega(\vec{p}) = \sqrt{\vec{p}^2 + M^2}. \tag{7}$$

For the discussion of κ-deformed free fields we introduce the operator-valued Weyl homomorphism

$$\hat{\varphi}(\hat{x}) \leftrightarrow \hat{\varphi}(x), \qquad \hat{\varphi}(\hat{y}) \leftrightarrow \hat{\varphi}(y), \tag{8}$$

where the relations (1) describing the pair of noncommutative κ-Minkowski points $\hat{x}$, $\hat{y}$ are supplemented by the following additional cross-relations[3]

$$[\hat{x}_0, \hat{y}_i] = \frac{\mathrm{i}}{\kappa}\hat{y}_i, \qquad [\hat{y}_0, \hat{x}_i] = \frac{\mathrm{i}}{\kappa}\hat{x}_i. \tag{9}$$

The corresponding κ-star multiplication prescription which represents the noncommutative space-time structure (1) and (9) looks as follows:

$$\hat{\varphi}(\hat{x}) \cdot \hat{\varphi}(\hat{y}) \leftrightarrow \hat{\varphi}(x) \star_\kappa \hat{\varphi}(y) = \frac{1}{(2\pi)^3} \int \mathrm{d}^4 p \int \mathrm{d}^4 q\, \mathrm{e}^{\mathrm{i}(p_0 x_0 + q_0 y_0) + (p_i \mathrm{e}^{\frac{q_0}{2\kappa}} x^i + q_i \mathrm{e}^{-\frac{p_0}{2\kappa}} y^i)}$$
$$\times \hat{A}(p_0, \vec{p})\hat{A}(q_0, \vec{q})\delta\big(C_\kappa^2(\vec{p}, p_0) - M^2\big)\delta\big(C_\kappa^2(\vec{q}, q_0) - M^2\big). \tag{10}$$

The formula (10) follows from the κ-star products of exponentials (3)

$$\mathopen{:}\mathrm{e}^{\mathrm{i}p_\mu \hat{x}^\mu}\mathclose{:} \cdot \mathopen{:}\mathrm{e}^{\mathrm{i}q_\mu \hat{y}^\mu}\mathclose{:} \leftrightarrow \mathrm{e}^{\mathrm{i}p_\mu x^\mu} \star_\kappa \mathrm{e}^{\mathrm{i}q_\mu y^\mu} = \mathrm{e}^{\mathrm{i}(p_0 x^0 + q_0 y^0) + (p_i \mathrm{e}^{\frac{q_0}{2\kappa}} x^i + q_i \mathrm{e}^{-\frac{p_0}{2\kappa}} y^i)}, \tag{11}$$

where the rhs of (11) is determined by the 4-momentum coproduct

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0, \qquad \Delta(P_i) = P_i \otimes \mathrm{e}^{\frac{P_0}{2\kappa}} + \mathrm{e}^{-\frac{P_0}{2\kappa}} \otimes P_i. \tag{12}$$

The field oscillators in the product $\hat{\varphi}(x) \star_\kappa \hat{\varphi}(y)$ (see (10)) carry the 4-momenta satisfying the standard κ-deformed mass-shell condition (5). We recall that in the models of κ-statistics presented in [3]–[5] such a standard κ-deformed mass-shell condition is imposed.

In our papers [2, 25] we modified the standard κ-star product (10) by the following additional deformation of the product of mass-shell deltas

$$\delta\big(C_\kappa^2(\vec{p}, p_0) - M^2\big)\delta\big(C_\kappa^2(\vec{q}, q_0) - M^2\big) \Rightarrow \delta\big(C_\kappa^2(\vec{p}\mathrm{e}^{\frac{q_0}{2\kappa}}, p_0) - M^2\big)\delta\big(C_\kappa^2(\vec{q}\mathrm{e}^{-\frac{p_0}{2\kappa}}, q_0) - M^2\big). \tag{13}$$

Such modification can also be introduced as the change of standard multiplication of the fields on noncommutative space time

$$\hat{\varphi}(\hat{x}) \cdot \hat{\varphi}(\hat{y}) \rightarrow \hat{\varphi}(\hat{x}) \cdot_\kappa \hat{\varphi}(\hat{y}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^4 p \int \mathrm{d}^4 q\, \hat{A}(p_0, \vec{p})\hat{A}(q_0, \vec{q})\mathrm{e}^{\mathrm{i}(p\hat{x} + q\hat{y})}$$
$$\times \delta\big(C_\kappa^2(\vec{p}\mathrm{e}^{\frac{q_0}{2\kappa}}, p_0) - M^2\big)\delta\big(C_\kappa^2(\vec{q}\mathrm{e}^{-\frac{p_0}{2\kappa}}, q_0) - M^2\big), \tag{14}$$

which is homomorphic to the following modified κ-star product of two κ-deformed free fields on the commuting Minkowski space

$$\hat{\varphi}(x)\hat{\star}_\kappa\hat{\varphi}(y) = \frac{1}{(2\pi)^3} \int \mathrm{d}^4 p \int \mathrm{d}^4 q\, \hat{A}(p_0, \vec{p})\hat{A}(q_0, \vec{q})\mathrm{e}^{\mathrm{i}(p_0 x_0 + q_0 y_0) + (p_i \mathrm{e}^{\frac{q_0}{2\kappa}} x^i + q_i \mathrm{e}^{-\frac{p_0}{2\kappa}} y^i)}$$
$$\times \delta\big(C_\kappa^2(\vec{p}\mathrm{e}^{\frac{q_0}{2\kappa}}, p_0) - M^2\big)\delta\big(C_\kappa^2(\vec{q}\mathrm{e}^{-\frac{p_0}{2\kappa}}, q_0) - M^2\big). \tag{15}$$

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[3] The relation (9) permits us to perform in consistency with (1) the limit $\hat{x}_\mu \rightarrow \hat{y}_\mu$. Another option for constructing the κ-deformed field theory is to assume that $[\hat{x}_\mu, \hat{y}_\nu] = 0$ (see e.g. [27]).

1.1.2. κ-deformation of the field oscillators algebra. We see from (15) that the field oscillators in the product $\hat{\varphi}(x)\hat{\star}_\kappa\hat{\varphi}(y)$ are put on modified mass shells (see (13)). The unconventional feature of such an approach is the use of quantized field oscillators $\hat{A}(p_0, \vec{p})$ extended to the values of $p = (p_0, \vec{p})$ which do not satisfy the κ-deformed mass-shell condition (4)

$$\hat{A}(p_0, \vec{p})|_{C_\kappa^2(\vec{p}, p_0)=M^2} \Rightarrow \hat{A}(p_0, \vec{p})|_{C_\kappa^2(\vec{p}, p_0)\neq M^2}. \tag{16}$$

The pair of modified mass-shell conditions following from (13) is

$$C_\kappa^2\left(\vec{p}\,e^{\frac{q_0}{2\kappa}}, p_0\right) - M^2 = 0, \qquad C_\kappa^2\left(\vec{q}\,e^{-\frac{p_0}{2\kappa}}, q_0\right) - M^2 = 0. \tag{17}$$

Such a modification can be interpreted by the coproduct relation (12), where $(p_0, \vec{p}\,e^{q_0/2\kappa})$ and $(q_0, \vec{q}\,e^{-p_0/2\kappa})$ correspond to first and second terms in the 4-momentum addition formula for the 2-particle state

$$P_0 \triangleright \hat{A}(p)\hat{A}(q) = (p_0 + q_0)\hat{A}(p)\hat{A}(q), \tag{18}$$

$$P_i \triangleright \hat{A}(p)\hat{A}(q) := m \circ [\Delta(P_i)\hat{A}(p) \otimes \hat{A}(q)] = [p_i e^{q_0/2\kappa} + q_i e^{-p_0/2\kappa}]\hat{A}(p)\hat{A}(q). \tag{19}$$

where $m \circ (f \otimes g) = fg$. The energy values p_0, q_0 satisfying (17) are equivalently described as the solutions $p_0^{(\epsilon,\epsilon')}(\vec{p}, \vec{q}), q_0^{(\epsilon,\epsilon')}(\vec{p}, \vec{q})$ ($\epsilon = \pm 1, \epsilon' = \pm 1$) of the following two coupled equations

$$p_0^{(\epsilon,\epsilon')} = \epsilon\omega_\kappa\left(\vec{p}\,e^{q_0^{(\epsilon,\epsilon')}/2\kappa}\right), \qquad q_0^{(\epsilon,\epsilon')} = \epsilon'\omega_\kappa\left(\vec{q}\,e^{-p_0^{(\epsilon,\epsilon')}/2\kappa}\right). \tag{20}$$

We see from (20) that for κ-deformed 2-particle states the energy of the first particle depends also on the 3-momenta of the energy, i.e. the modification (13) of the mass-shell condition couples both constituents of the 2-particle state.

For a particular choice of the κ-deformed oscillator algebra we did show in [2, 25] that by using the modified κ-star product (15) one obtains the *c*-number field commutator. One obtains

$$[\hat{\varphi}(x), \hat{\varphi}(y)]_{\hat{\star}_\kappa} = i\Delta_\kappa(x - y; M^2), \tag{21}$$

where

$$\Delta_\kappa(x; M^2) = \frac{i}{(2\pi)^3} \int d^4 p \,\epsilon(p_0)\delta\left((2\kappa \sinh(p_0/2\kappa))^2 - \vec{p}^2 - M^2\right) e^{i p_\mu x^\mu} \tag{22}$$

is the κ-deformed Pauli–Jordan commutator function proposed first in [12] after using somewhat naive arguments.

1.2. Plan of the paper and the resume of results

In our earlier papers [1, 2, 25] we studied a definite model of κ-deformed oscillators, without any free parameter (except the deformation parameter κ). In this paper we consider the most general set of binary κ-deformed oscillator algebras consistent with the κ-deformed addition law of the 4-momenta.

First, in section 2 we shall use the κ-star product (10) and the oscillators $\hat{A}(p_0, \vec{p})$ which lie on standard κ-deformed mass shells. We shall show that in such a case for any possible choice of a binary oscillator algebra it is not possible to obtain the *c*-number value of the field commutator. In place of formula (21) one gets the *q*-number field commutator bilinear in the κ-deformed oscillators. In particular we shall discuss briefly two recent proposals of κ-deformed oscillator algebras ([3, 4]) which fall into such a category.

In section 3 we shall consider the binary multiplication of κ-deformed free fields using the modified κ-deformed ⋆-product. In such a case by performing the general transformation

in 2-particle 4-momentum space we arrive at a large class of κ-deformed oscillator algebras depending on six arbitrary functions of 2-particle 4-momenta, which all lead to the *c*-number field commutator. All such κ-oscillators are characterized by the modified energy–momentum dispersion relation generalizing the relations (17) or (20) and can be classified by various forms of the addition law for the 3-momenta of κ-deformed 2-particle state. The particular choice proposed in [1, 2] was described by the Abelian addition law

$$\vec{p}_{1+2} = \vec{p} + \vec{q}, \tag{23}$$

while the choices from [3, 4] (see also [16]) were characterized by the non-Abelian addition formula[4]

$$\vec{p}_{1+2} = \vec{p} \dotplus \vec{q} = \vec{p}\,\mathrm{e}^{\frac{q_0}{2\kappa}} + \vec{q}\,\mathrm{e}^{-\frac{p_0}{2\kappa}}. \tag{24}$$

The κ-deformed oscillator algebra with the 3-momentum addition law (24) was constructed by the use of suitable deformation of the flip operator. In section 4 we describe the general oscillator algebras (61) (see section 3) as the composition of the most general κ-deformed multiplication and the κ-deformed flip operation.

The κ-deformed field oscillators determine the structure of the corresponding κ-deformed multi-particle states[5]. We shall only mention here that the *n*-particle sector of κ-deformed Fock space is represented by suitably constructed non-factorizable clusters. We interpret such a structure of κ-deformed Fock spaces as a result of the geometric interactions implied by the Lie-algebraic noncommutativity of κ-deformed space time. It would be interesting to understand how such features can be linked with the quantum gravity framework.

2. The standard κ-star product and the κ-deformed quantum fields

The aim of this section is to study the commutator of the fields (2) with the κ-star multiplication rule (10)

$$
\begin{aligned}
[\hat{\varphi}(x), \hat{\varphi}(y)]_{\star_\kappa} ={}& \frac{1}{(2\pi)^3} \int \mathrm{d}^4 p\, \mathrm{d}^4 q\, \hat{A}(p)\hat{A}(q)\mathrm{e}^{\mathrm{i}px} \star_\kappa \mathrm{e}^{\mathrm{i}qy} \delta\big(C_\kappa^2(p) - M^2\big)\delta\big(C_\kappa^2(q) - M^2\big) \\
&- \frac{1}{(2\pi)^3} \int \mathrm{d}^4 p'\mathrm{d}^4 q'\, \hat{A}(q')\hat{A}(p')\mathrm{e}^{\mathrm{i}q'y} \star_\kappa \mathrm{e}^{\mathrm{i}p'x} \delta\big(C_\kappa^2(p') - M^2\big)\delta\big(C_\kappa^2(q') - M^2\big).
\end{aligned}
\tag{25}
$$

We see from (25) that due to the presence of respective Dirac deltas the field oscillators remain on the κ-deformed mass shell (4). We shall look for the generalized binary relations of κ-deformed oscillators consistent with the κ-deformed 4-momentum addition law. We shall show that for any choice of these binary relations determining the choice of κ-statistics, the commutator (25) is an operator bilinear in the field oscillators. We use in this paper the definition of a (generalized) free field as characterized by the *c*-number commutator function (see e.g. [23, 24]). We obtain therefore in this section the result that by using standard on-shell κ-oscillators $\hat{A}(p_0, \vec{p})$ we can not obtain the free κ-deformed quantum fields.

Our demonstration of the operator nature of the commutator (25) follows from the impossibility of the factorization under momenta integrals of any binary relation for the field oscillators $\hat{A}(p_0, \vec{p})$. For studying the possible factorization we shall perform the following general O(3)-covariant change of variables separately in the first term on the rhs of (25) ($\tilde{f} = \tilde{f}(p, q)$, etc):

$$\vec{p} \rightarrow \vec{\mathcal{P}}(p, q) = \vec{p}\tilde{f} + \vec{q}\tilde{g}, \qquad \vec{q} \rightarrow \vec{\mathcal{Q}}(p, q) = \vec{p}\tilde{k} + \vec{q}\tilde{l}, \tag{26}$$

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[4] We recall that we use the κ-Poincaré Hopf algebra which leads to the standard (see [11]) coproduct (12) for the 3-momenta.

[5] The discussion of multi-particle states for $n > 2$ has been recently considered in [28].

$$p_0 \to \mathcal{P}_0(p, q), \qquad q_0 \to \mathcal{Q}_0(p, q), \tag{27}$$

with the following inverse formulae ($f = f(\mathcal{P}, \mathcal{Q})$, etc)

$$\vec{\mathcal{P}} \to \vec{p}(\mathcal{P}, \mathcal{Q}) = \vec{\mathcal{P}} f + \vec{\mathcal{Q}} g, \qquad \vec{\mathcal{Q}} \to \vec{q}(\mathcal{P}, \mathcal{Q}) = \vec{\mathcal{P}} k + \vec{\mathcal{Q}} l, \tag{28}$$

$$\mathcal{P}_0 \to p_0(\mathcal{P}, \mathcal{Q}), \qquad \mathcal{Q}_0 \to q_0(\mathcal{P}, \mathcal{Q}), \tag{29}$$

and in the second term on the rhs of (25)

$$\vec{p}' \to \vec{\mathcal{P}}' = \vec{p}' \tilde{f}' + \vec{q}' \tilde{g}', \qquad \vec{q}' \to \vec{\mathcal{Q}}' = \vec{p}' \tilde{k}' + \vec{q}' \tilde{l}', \tag{30}$$

$$p_0' \to \mathcal{P}_0', \qquad q_0' \to \mathcal{Q}_0', \tag{31}$$

with the inverse formulae

$$\vec{\mathcal{P}}' \to \vec{p}'(\mathcal{P}', \mathcal{Q}') = \vec{\mathcal{P}}' f' + \vec{\mathcal{Q}}' g', \qquad \vec{\mathcal{Q}}' \to \vec{q}'(\mathcal{P}', \mathcal{Q}') = \vec{\mathcal{P}}' k' + \vec{\mathcal{Q}}' l', \tag{32}$$

$$\mathcal{P}_0 \to p_0'(\mathcal{P}', \mathcal{Q}'), \qquad \mathcal{Q}_0 \to q_0'(\mathcal{P}', \mathcal{Q}'). \tag{33}$$

We assume that f, g, h, k, p_0, q_0 in (28) and $f', g', h', k', p_0', q_0'$ in (32) are respectively arbitrary O(3)-invariant functions of the 4-momenta $\mathcal{P} = (\vec{\mathcal{P}}, \mathcal{P}_0)$, $\mathcal{Q} = (\vec{\mathcal{Q}}, \mathcal{Q}_0)$ and $\mathcal{P}' = (\vec{\mathcal{P}}', \mathcal{P}_0')$, $\mathcal{Q}' = (\vec{\mathcal{Q}}', \mathcal{Q}_0')$.[6]

We shall look for such a choice of arbitrary functions in formulae (28), (29) and (32), (33) which leads to the equality

$$e^{ip(\mathcal{P}, \mathcal{Q})x} \star_\kappa e^{iq(\mathcal{P}, \mathcal{Q})y} = e^{iq'(\mathcal{P}, \mathcal{Q})y} \star_\kappa e^{ip'(\mathcal{P}, \mathcal{Q})x}, \tag{34}$$

with κ-deformed mass-shell conditions taken into account. Formula (34) describes a necessary condition which permits us to factorize in the commutator (25) the oscillator algebra and to derive the *c*-number commutator function. We should observe that the change of variables (26)–(32) modifies the explicit form of the product of mass-shell deltas in (25) as well. We obtain in the first term of the rhs of (25)

$$\delta\big(C_\kappa^2(p) - M^2\big) \cdot \delta\big(C_\kappa^2(q) - M^2\big)$$
$$\to \delta\big(C_\kappa^2(p(\mathcal{P}, \mathcal{Q})) - M^2\big) \cdot \delta\big(C_\kappa^2(q(\mathcal{P}, \mathcal{Q})) - M^2\big), \tag{35}$$

and in the second term we obtain

$$\delta\big(C_\kappa^2(p') - M^2\big) \cdot \delta\big(C_\kappa^2(q') - M^2\big)$$
$$\to \delta\big(C_\kappa^2(p'(\mathcal{P}, \mathcal{Q})) - M^2\big) \cdot \delta\big(C_\kappa^2(q'(\mathcal{P}, \mathcal{Q})) - M^2\big). \tag{36}$$

One obtains[7]

$$[\hat{\varphi}(x), \hat{\varphi}(y)]_{\star_\kappa} = \frac{1}{(2\pi)^3} \int d^4\mathcal{P} \, d^4\mathcal{Q} J\left(\overset{p,q}{\mathcal{P},\mathcal{Q}}\right) \hat{A}(p(\mathcal{P}, \mathcal{Q})) \hat{A}(q(\mathcal{P}, \mathcal{Q}))$$
$$\cdot e^{ip(\mathcal{P}, \mathcal{Q})x} \star_\kappa e^{iq(\mathcal{P}, \mathcal{Q})y}$$
$$\cdot \delta\big(C_\kappa^2(p(\mathcal{P}, \mathcal{Q})) - M^2\big) \cdot \delta\big(C_\kappa^2(q(\mathcal{P}, \mathcal{Q})) - M^2\big)$$
$$- \frac{1}{(2\pi)^3} \int d^4\mathcal{P} \, d^4\mathcal{Q} J\left(\overset{p',q'}{\mathcal{P},\mathcal{Q}}\right) \hat{A}(q'(\mathcal{P}, \mathcal{Q})) \hat{A}(p'(\mathcal{P}, \mathcal{Q}))$$
$$\cdot e^{iq'(\mathcal{P}, \mathcal{Q})y} \star_\kappa e^{ip'(\mathcal{P}, \mathcal{Q})x}$$
$$\cdot \delta\big(C_\kappa^2(p'(\mathcal{P}, \mathcal{Q})) - M^2\big) \cdot \delta\big(C_\kappa^2(q'(\mathcal{P}, \mathcal{Q})) - M^2\big). \tag{37}$$

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[6] We recall that the classical SO(3) Hopf algebra is a sub-Hopf algebra of the κ-deformed Poincaré Hopf algebra.

[7] Because in transformed formula (25) we integrate over the variables $\mathcal{P}', \mathcal{Q}'$, further we shall denote them similarly as in the first term of the commutator by $\mathcal{P}$ and $\mathcal{Q}$.

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Let us denote by $\mathcal{P}_0 = \pi(\vec{\mathcal{P}}, \vec{\mathcal{Q}})$, $\mathcal{Q}_0 = \rho(\vec{\mathcal{P}}, \vec{\mathcal{Q}})$ and $\mathcal{P}'_0 = \pi'(\vec{\mathcal{P}}, \vec{\mathcal{Q}})$, $\mathcal{Q}'_0 = \rho'(\vec{\mathcal{P}}, \vec{\mathcal{Q}})$ the solutions of the following two pairs of the deformed mass-shell conditions

$$C_\kappa^2(p(\mathcal{P}, \mathcal{Q})) - M^2 = 0, \qquad C_\kappa^2(q(\mathcal{P}, \mathcal{Q})) - M^2 = 0, \tag{38}$$

$$C_\kappa^2(p'(\mathcal{P}, \mathcal{Q})) - M^2 = 0, \qquad C_\kappa^2(q'(\mathcal{P}, \mathcal{Q})) - M^2 = 0. \tag{39}$$

We shall consider first the validity of restricted relation (34) obtained by putting $x_0 = y_0 = 0$. It is easy to check that one gets the equality of deformed 3-momenta exponentials

$$e^{ip(\mathcal{P},\mathcal{Q})x} \star_\kappa e^{iq(\mathcal{P},\mathcal{Q})y}|_{\mathcal{P}_0=\pi,\mathcal{Q}_0=\rho,(x_0=y_0=0)}$$

$$= \exp\left[i\left(\vec{p}(\mathcal{P}, \mathcal{Q})e^{\frac{q_0(\mathcal{P},\mathcal{Q})}{2\kappa}}\vec{x} + \vec{q}(\mathcal{P}, \mathcal{Q})e^{-\frac{p_0(\mathcal{P},\mathcal{Q})}{2\kappa}}\vec{y}\right)\right]|_{\mathcal{P}_0=\pi,\mathcal{Q}_0=\rho}$$

$$= \exp\left[i\left(\vec{q}'(\mathcal{P}, \mathcal{Q})e^{\frac{p'_0(\mathcal{P},\mathcal{Q})}{2\kappa}}\vec{y} + \vec{p}'(\mathcal{P}, \mathcal{Q})e^{-\frac{q'_0(\mathcal{P},\mathcal{Q})}{2\kappa}}\vec{x}\right)\right]|_{\mathcal{P}_0=\pi',\mathcal{Q}_0=\rho'}$$

$$= e^{iq'(\mathcal{P},\mathcal{Q})y} \star_\kappa e^{ip'(\mathcal{P},\mathcal{Q})x}|_{\mathcal{P}'_0=\pi',\mathcal{Q}'_0=\rho',(x_0=y_0=0)}, \tag{40}$$

if the arbitrary functions introduced in relations (28), (29) and (32), (33) satisfy the relations

$$f(\mathcal{P}, \mathcal{Q})e^{p_0(\mathcal{P},\mathcal{Q})/2\kappa} = f'(\mathcal{P}, \mathcal{Q})e^{-p'_0(\mathcal{P},\mathcal{Q})/2\kappa}, \quad g(\mathcal{P}, \mathcal{Q})e^{p_0(\mathcal{P},\mathcal{Q})/2\kappa} = g'(\mathcal{P}, \mathcal{Q})e^{-p'_0(\mathcal{P},\mathcal{Q})/2\kappa},$$

$$k(\mathcal{P}, \mathcal{Q})e^{-q_0(\mathcal{P},\mathcal{Q})/2\kappa} = k'(\mathcal{P}, \mathcal{Q})e^{q'_0(\mathcal{P},\mathcal{Q})/2\kappa}, \quad l(\mathcal{P}, \mathcal{Q})e^{-q_0(\mathcal{P},\mathcal{Q})/2\kappa} = l'(\mathcal{P}, \mathcal{Q})e^{q'_0(\mathcal{P},\mathcal{Q})/2\kappa},$$

$$\tag{41}$$

and $\mathcal{P}_0 = \pi$, $\mathcal{Q}_0 = \rho$, $\mathcal{P}_0 = \pi'$, $\mathcal{Q}_0 = \rho'$ satisfy the mass-shell conditions (38) and (39).

If we assume relations (41), then formula (34) reduces to the equality of time exponentials

$$e^{i[x_0 p_0(\mathcal{P},\mathcal{Q})+y_0 q_0(\mathcal{P},\mathcal{Q})]}|_{\mathcal{P}_0=\pi,\mathcal{Q}_0=\rho} = e^{i[x_0 p'_0(\mathcal{P},\mathcal{Q})+y_0 q'_0(\mathcal{P},\mathcal{Q})]}|_{\mathcal{P}_0=\pi',\mathcal{Q}_0=\rho'}. \tag{42}$$

Relation (42) is satisfied for any value of x_0, y_0 only if

$$p_0(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho) = p'_0(\vec{\mathcal{P}}, \pi'; \vec{\mathcal{Q}}, \rho'), \quad q_0(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho) = q'_0(\vec{\mathcal{P}}, \pi'; \vec{\mathcal{Q}}, \rho'). \tag{43}$$

In order to derive the restrictions following from (43) one can rewrite the relations (38) and (39) in the form of the following identities ($\epsilon = \pm 1$, etc)

$$p_0(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho) = \epsilon\omega_\kappa(\vec{p}^2(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho)), \quad q_0(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho) = \eta\omega_\kappa(\vec{q}^2(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho)), \tag{44}$$

$$p'_0(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho) = \epsilon'\omega_\kappa(\vec{p}'^2(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho)), \quad q'_0(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho) = \eta'\omega_\kappa(\vec{q}'^2(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho)), \tag{45}$$

where $\omega_\kappa(\vec{p})$ is defined by relation (7). We see from (44) and (45) that relation (43) can be written as follows:[8]

$$\omega_\kappa(\vec{p}^2(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho)) = \omega_\kappa(\vec{p}'^2(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho)), \tag{46}$$

$$\omega_\kappa(\vec{q}^2(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho)) = \omega_\kappa(\vec{q}'^2(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho)), \tag{47}$$

which implies that

$$\vec{p}^2(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho) = \vec{p}'^2(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho), \qquad \vec{q}^2(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho) = \vec{q}'^2(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho). \tag{48}$$

Inserting in the first formula of (48) the relations (28), (29) and (32), (33) one gets from (46) the condition

$$f^2\vec{\mathcal{P}}^2 + 2fg\vec{\mathcal{P}}\vec{\mathcal{Q}} + g^2\vec{\mathcal{Q}}^2 = f'^2\vec{\mathcal{P}}^2 + 2f'g'\vec{\mathcal{P}}\vec{\mathcal{Q}} + g'^2\vec{\mathcal{Q}}^2, \tag{49}$$

and analogous relation obtained by the replacements $\vec{p} \to \vec{p}'$ and $\vec{q} \to \vec{q}'$ in (47). The insertion of (41) into (49) leads to

$$(1 - e^{(p_0+p'_0)/2\kappa})\vec{p}^2(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho) = 0. \tag{50}$$

---

[8] Because $\omega_\kappa \geqslant 0$ the relations (43) require that $\epsilon = \epsilon'$ and $\eta = \eta'$.

Because $\vec{p}^2$ is positive definite, condition (49) requires for finite κ that $p_0(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho) = -p_0'(\vec{\mathcal{P}}, \pi'; \vec{\mathcal{Q}}, \rho')$, $q_0(\vec{\mathcal{P}}, \pi; \vec{\mathcal{Q}}, \rho) = -q_0'(\vec{\mathcal{P}}, \pi'; \vec{\mathcal{Q}}, \rho')$ which contradicts the relations (43).

In conclusion, both relations (41) and (43) required for the validity of the relation (34) cannot be valid, and consequently if $x_0 \neq y_0$ for on-shell oscillators, it is not possible to factorize in the κ-deformed commutator function (25) any κ-deformed oscillator algebra.

Examples in recent literature. In recent papers [3]–[5] the following particular choice has been made:

$$\vec{p} = \vec{\mathcal{P}}(f = 1, g = 0), \quad \vec{q} = \vec{\mathcal{Q}}(l = 1, k = 0), \quad p_0 = \mathcal{P}_0, \quad q_0 = \mathcal{Q}_0. \quad (51)$$

The restrictions on functions $\vec{p}'(\mathcal{P}, \mathcal{Q})$ and $\vec{q}'(\mathcal{P}, \mathcal{Q})$ were obtained from the non-Abelian composition law of 3-momenta (see (24)), considered as an identity in the variables $\vec{\mathcal{P}}$ and $\vec{\mathcal{Q}}$,

$$\vec{\mathcal{P}} \dotplus \vec{\mathcal{Q}} = \vec{q}'(\mathcal{P}, \mathcal{Q}) \dotplus \vec{p}'(\mathcal{P}, \mathcal{Q}), \quad (52)$$

where $\dotplus$ denotes the addition law based on the κ-deformed 4-momentum coproduct. The second relation follows from the energy conservation law.

For our choice of the 3-momenta coproduct (see (12)) equation (52) takes the following explicit form

$$\vec{\mathcal{P}}\mathrm{e}^{\frac{\mathcal{P}_0}{2\kappa}} + \vec{\mathcal{Q}}\mathrm{e}^{-\frac{\mathcal{Q}_0}{2\kappa}} = \vec{q}'(\mathcal{P}, \mathcal{Q})\mathrm{e}^{\frac{p_0'}{2\kappa}} + \vec{p}'(\mathcal{P}, \mathcal{Q})\mathrm{e}^{-\frac{q_0'}{2\kappa}}, \quad (53)$$

where $\mathcal{P}_0 = \pm\omega_\kappa(\vec{\mathcal{P}})$, $\mathcal{Q}_0 = \pm\omega_\kappa(\vec{\mathcal{Q}})$ and the relations (45) should be inserted on the rhs of (53). One gets the relation (53) as identity in particular if[9]

$$\vec{p}'(\mathcal{P}, \mathcal{Q}) = \vec{\mathcal{P}} \exp\left(\frac{\mathcal{P}_0 + q_0'}{2\kappa}\right), \quad \vec{q}'(\mathcal{P}, \mathcal{Q}) = \vec{\mathcal{Q}} \exp\left(-\frac{\mathcal{Q}_0 + p_0'}{2\kappa}\right), \quad (54)$$

with the relations (45) taking the following explicit form ($\epsilon' = \pm1$, $\eta' = \pm1$)

$$p_0' = \epsilon'\omega_\kappa\left(\vec{\mathcal{P}}\mathrm{e}^{\frac{\mathcal{P}_0 + q_0'}{2\kappa}}\right), \qquad q_0' = \eta'\omega_\kappa\left(\vec{\mathcal{Q}}\mathrm{e}^{-\frac{(\mathcal{Q}_0 + p_0')}{2\kappa}}\right), \quad (55)$$

determining $p_0' = p_0'(\vec{\mathcal{P}}, \vec{\mathcal{Q}})$ and $q_0' = q_0'(\vec{\mathcal{P}}, \vec{\mathcal{Q}})$ as functions of 3-momenta $\vec{\mathcal{P}}$ and $\vec{\mathcal{Q}}$. The energy conservation relation takes the form

$$\mathcal{P}_0 + \mathcal{Q}_0 = p_0' + q_0' \Leftrightarrow \epsilon\omega_\kappa(\vec{\mathcal{P}}) + \eta\omega_\kappa(\vec{\mathcal{Q}}) = \epsilon'\omega_\kappa\left(\vec{\mathcal{P}}\mathrm{e}^{\frac{\mathcal{P}_0 + q_0'}{2\kappa}}\right) + \eta'\omega_\kappa\left(\vec{\mathcal{Q}}\mathrm{e}^{-\frac{(\mathcal{Q}_0 + p_0')}{2\kappa}}\right). \quad (56)$$

It should be stressed however that even if the relations (56) are valid the relations (43) crucial for obtaining the c-number commutator cannot be satisfied.

One can point out that equations (53) do not specify completely the six functions $p_\mathrm{i}'(\mathcal{P}, \mathcal{Q}), q_\mathrm{i}'(\mathcal{P}, \mathcal{Q})$. In [4, 5] it has been additionally assumed that two products of the oscillators ($\hat{A}(\vec{\mathcal{P}})\hat{A}(\vec{\mathcal{Q}})$ and multiplied in flipped order $\hat{A}(\vec{q}', q_0')\hat{A}(\vec{p}', p_0')$) transform in the same covariant way under the κ-deformed boost generators. It was shown that

(i) for the $D = 2$ κ-deformed system, there exists a unique κ-covariant choice of functions $\vec{p}', \vec{q}'$ consistent with relations (56) and the $D = 2$ counterpart of (53),

(ii) in $D = 4$ under analogous assumptions the solution is known only in the lowest three orders of the $\frac{1}{\kappa}$ perturbation expansion.

The results obtained in [4, 5] providing the κ-covariant set of κ-statistics are very interesting, but because they describe the states with the 4-momenta satisfying standard on-shell conditions (38) and (39) they are not consistent with the relation (43) which is necessary for obtaining the field commutator as a c-number.

---

[9] In [3]–[5] a different bi-crossproduct basis of the κ-deformed Poincaré algebra is used, which leads to the coproduct $\Delta(\vec{P}') = \vec{P}' \otimes 1 + \mathrm{e}^{-\frac{P_0}{\kappa}} \otimes \vec{P}'$ and suitable modification of our formulae (53) and (54).

3. Modified κ-deformed star product and free κ-deformed quantum fields

We see from section 2 that the main problem in obtaining c-number field commutators is the difficulty with getting valid relations (43). We recall that in section 2 we obtained different on-shell values of $p_0(\mathcal{P}, \mathcal{Q})$, $q_0(\mathcal{P}, \mathcal{Q})$ and $p'_0(\mathcal{P}, \mathcal{Q})$, $q'_0(\mathcal{P}, \mathcal{Q})$ following from different forms of the κ-deformed mass-shell conditions (38) and (39). In this section we shall change the κ-deformed star product (10) in such a way that both modified mass-shell conditions (38) and (39) will become identical. We will provide two pairs of the same modified energy–momentum dispersion relations for $p_0(\mathcal{P}, \mathcal{Q})$, $p'_0(\mathcal{P}, \mathcal{Q})$ and $q_0(\mathcal{P}, \mathcal{Q})$, $q'_0(\mathcal{P}, \mathcal{Q})$, which allows the validity of relations (43) and the c-number commutation function.

For that purpose we shall use a modified κ-deformed star product (15). The corresponding field commutator of κ-deformed free fields (2) takes the form

$$
\begin{aligned}
[\hat{\varphi}(x), \hat{\varphi}(y)]_{\hat{\star}_\kappa} = {}& \frac{1}{(2\pi)^3} \int d^4 p \, d^4 q \, \hat{A}(p) \hat{A}(q) e^{ipx} \star_\kappa e^{iqy} \\
& \cdot \delta\big(C_\kappa^2(\vec{p}e^{q_0/2\kappa}, p_0) - M^2\big) \delta(C^2(\vec{q}e^{-p_0/2\kappa}, q_0) - M^2) \\
& - \frac{1}{(2\pi)^3} \int d^4 p' d^4 q' \, \hat{A}(q') \hat{A}(p') e^{iq'y} \star_\kappa e^{ip'x} \\
& \cdot \delta\big(C_\kappa^2(q'_0, \vec{q'}e^{p'_0/2\kappa}) - M^2\big) \delta(C^2(p'_0, \vec{p'}e^{-q'_0/2\kappa}) - M^2).
\end{aligned}
\tag{57}
$$

Introducing the change of momentum variables (26)–(33), one obtains the formula

$$
\begin{aligned}
[\hat{\varphi}(x), \hat{\varphi}(y)]_{\hat{\star}_\kappa} = {}& \frac{1}{(2\pi)^3} \int d^4 \mathcal{P} \, d^4 \mathcal{Q} \, J\left(\overset{p,q}{\mathcal{P}, \mathcal{Q}}\right) \hat{A}(p_0, \vec{\mathcal{P}}f + \vec{\mathcal{Q}}g) \hat{A}(q_0, \vec{\mathcal{P}}k + \vec{\mathcal{Q}}l) \\
& \cdot \exp\left[i(p_0 x^0 + q_0 y^0)\right] \exp\left[-i[(\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g)e^{q_0/2\kappa}\vec{x} + (\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l)e^{-p_0/2\kappa}\vec{y}]\right] \\
& \cdot \delta\big(C_\kappa^2(p_0, [\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g]e^{q_0/2\kappa}) - M^2\big) \delta\big(C_\kappa^2(q_0, [\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l]e^{-p_0/2\kappa}) - M^2\big) \\
& - \frac{1}{(2\pi)^3} \int d^4 \mathcal{P} d^4 \mathcal{Q} \, J\left(\overset{p',q'}{\mathcal{P}, \mathcal{Q}}\right) \hat{A}(q'_0, \vec{\mathcal{P}}k' + \vec{\mathcal{Q}}l') \hat{A}(p'_0, \vec{\mathcal{P}}f' + \vec{\mathcal{Q}}g') \\
& \cdot \exp i(p'_0 x^0 + q'_0 y^0) \exp[-i[(\vec{\mathcal{P}}k' + \vec{\mathcal{Q}}l')e^{p'_0/2\kappa}\vec{y} + (\vec{\mathcal{P}}f' + \vec{\mathcal{Q}}g')e^{-q'_0/2\kappa}\vec{x}]] \\
& \cdot \delta\big(C_\kappa^2(q'_0, [\vec{\mathcal{P}}k' + \vec{\mathcal{Q}}l']e^{p'_0/2\kappa}) - M^2\big) \delta(C^2(p'_0, [\vec{\mathcal{P}}f' + \vec{\mathcal{Q}}g']e^{-q'_0/2\kappa}) - M^2),
\end{aligned}
\tag{58}
$$

where $J\big(\overset{p,q}{\mathcal{P}, \mathcal{Q}}\big)$ and $J\big(\overset{p',q'}{\mathcal{P}, \mathcal{Q}}\big)$ describe respectively the Jacobians of transformations (28), (29) and (32), (33). We see that by replacement (13) we matched in two consecutive terms in (57) the asymmetry of the star product of exponentials with the asymmetry of mass-shell deltas. After the substitution of relations (41) (expressing f', g', k', l', p'_0, q'_0 by f, g, k, l, p_0, q_0) we see that in (58) the products of two mass-shell deltas in two consecutive terms are becoming the same, and one can proceed to factorize the binary algebraic relations describing the κ-deformed oscillator algebra. After inserting relations (41) the energy values p_0, p'_0 and q_0, q'_0 will satisfy the same mass-shell conditions and therefore it will be consistent to assume relations (43).

Relations (43) provide necessary conditions for the factorization of the κ-deformed algebra. One obtains

$$
\begin{aligned}
[\hat{\varphi}(x), \hat{\varphi}(y)]_{\hat{\star}_\kappa} = {}& \int d^4 \mathcal{P} \, d^4 \mathcal{Q} \Bigg[J\left(\overset{p,q}{\mathcal{P}, \mathcal{Q}}\right) \hat{A}(p_0, \vec{\mathcal{P}}f + \vec{\mathcal{Q}}g) \hat{A}(q_0, \vec{\mathcal{P}}k + \vec{\mathcal{Q}}l) \\
& - J\left(\overset{p',q'}{\mathcal{P}, \mathcal{Q}}\right) \hat{A}(q_0, (\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l)e^{-p_0/\kappa}) \hat{A}(p_0, (\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g)e^{q_0/\kappa}) \Bigg] \\
& \cdot \exp\left[i(p_0 x^0 + q_0 y^0)\right] \exp(-i[(\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g)e^{q_0/2\kappa}\vec{x} + (\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l)e^{-p_0/2\kappa}\vec{y}]) \\
& \cdot \delta\big(C_\kappa^2(p_0, [\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g]e^{q_0/2\kappa}) - M^2\big) \delta\big(C_\kappa^2(q_0, [\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l]e^{-p_0/2\kappa}) - M^2\big).
\end{aligned}
\tag{59}
$$

Under the integral (59) $\mathcal{P}_0 \equiv \tilde{\pi}(\vec{\mathcal{P}}, \vec{\mathcal{Q}})$ and $\mathcal{Q}_0 \equiv \tilde{\rho}(\vec{\mathcal{P}}\vec{\mathcal{Q}})$ describe respectively the solutions of the following coupled pair of modified κ-deformed mass-shell conditions

$$C_\kappa^2(p_0, [\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g]e^{q_0/2\kappa}) - M^2 = 0, \qquad C_\kappa^2(q_0, [\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l]e^{-p_0/2\kappa}) - M^2 = 0. \qquad (60)$$

In order to obtain the *c*-number value of the commutator (58) we should postulate the following general κ-deformed oscillator algebra

$$J\left(\overset{p,q}{\mathcal{P},\mathcal{Q}}\right) \hat{A}(p_0, \vec{\mathcal{P}}f + \vec{\mathcal{Q}}g)\hat{A}(q_0, \vec{\mathcal{P}}k + \vec{\mathcal{Q}}l) \qquad (61)$$

$$- J\left(\overset{p',q'}{\mathcal{P},\mathcal{Q}}\right) \hat{A}(q_0, (\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l)e^{-p_0/\kappa})\hat{A}(p_0, (\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g)e^{q_0/\kappa}) = c - \text{number},$$

where the functions $p_0 = p_0(\tilde{\pi}, \vec{\mathcal{P}}; \tilde{\rho}, \vec{\mathcal{Q}})$, $q_0 = q_0(\tilde{\pi}, \vec{\mathcal{P}}; \tilde{\rho}, \vec{\mathcal{Q}})$ do satisfy the mass-shell equations (60). We add that the *c*-number on the rhs of (61) should be proportional to the Planck constant $\hbar$.

For classical κ-deformed fields the *c*-number on the rhs of the relation (61) vanishes and in such a case relations (61) describe the κ-deformed braided oscillators. Substituting into (59) relation (61) with a vanishing *c*-number one obtains the commutator of the κ-braided free fields, describing the κ-deformation of the classical fields,

$$[\hat{\varphi}^{\mathrm{cl}}(x), \hat{\varphi}^{\mathrm{cl}}(y)]_{\hat{\star}_\kappa} = 0. \qquad (62)$$

In quantum κ-deformed field theory the nonvanishing *c*-number on the rhs of equation (61) is proportional to Dirac delta with the argument determined by the κ-deformed 3-momentum addition law. Such a term in the general case is specified in the appendix (see also (67)).

The oscillators $\hat{A}(p_0, \vec{p})$ which are present in relation (61) carry the 4-momentum $p_\mu = (p_0, \vec{p})$

$$P_\mu \triangleright \hat{A}(p_0, \vec{p}) = p_\mu \hat{A}(p_0, \vec{p}), \qquad (63)$$

restricted by the modified κ-deformed mass-shell conditions (60). The product of two oscillators carry respectively the 4-momenta determined by the coproduct rule (12)

$$P_i \triangleright \left(\hat{A}(p_0, \vec{p})\hat{A}(q_0, \vec{q})\right) = \left(p_i e^{\frac{q_0}{2\kappa}} + q_i e^{-\frac{p_0}{2\kappa}}\right)\left(\hat{A}(p_0, \vec{p})\hat{A}(q_0, \vec{q})\right), \qquad (64)$$

$$P_0 \triangleright \left(\hat{A}(p_0, \vec{p})\hat{A}(q_0, \vec{q})\right) = (p_0 + q_0)\left(\hat{A}(p_0, \vec{p})\hat{A}(q_0, \vec{q})\right). \qquad (65)$$

Applying the rules (64) and (65) to both products of oscillators occurring in (61) we should obtain the same eigenvalues. We obtain the following relations:

(i) the class of addition laws for 3-momenta

$$\vec{p}^{(1+2)} = \left(\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g\right)e^{q_0/2\kappa} + \left(\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l\right)e^{-p_0/2\kappa} = \vec{p}^{(2+1)}, \qquad (66)$$

satisfied as identity for arbitrary values of $\vec{p}$ and $\vec{q}$. Such an additional law implies the following change of the 3-momentum Dirac delta in the oscillator algebra (see also (A.6) and (A.7) in the appendix)

$$\delta^{(3)}(\vec{\mathcal{P}} - \vec{\mathcal{Q}}) \rightarrow \delta^{(3)}\left([\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g]e^{q_0/2\kappa} - [\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l]e^{-p_0/2\kappa}\right), \qquad (67)$$

(ii) the standard addition law for energy

$$p_0^{(1+2)} = p_0 + q_0 = p_0^{(2+1)}. \qquad (68)$$

Relations (61) and (66) depend on four functions f, g, k, l; the values of p_0 and q_0 are determined from the pair of equations (60). Different choices of κ-deformed oscillator algebras (61) can be classified by the corresponding explicit form of 3-momentum addition laws described by (66). We add that relation (61) decomposes into four sets of bilinear relations for κ-deformed creation and annihilation operators (see the appendix).

In order to obtain explicit formulae we shall consider now the particular cases of general relation (61). First, we recall the algebraic scheme providing the Abelian addition law [1, 2] and further, we present the framework providing the c-number commutator in the case of the non-Abelian addition law (24) (see also [3]–[5]).

3.1. Abelian addition law [1, 2, 25]

In such a case the functions occurring in (61) are the following

$$f = \mathrm{e}^{-\frac{\mathcal{Q}_0}{2\kappa}}, \quad g = 0, \quad k = 0, \quad l = \mathrm{e}^{\frac{\mathcal{P}_0}{2\kappa}}; \quad p_0 = \mathcal{P}_0, \quad q_0 = \mathcal{Q}_0. \tag{69}$$

The mass-shell conditions (60) take the form

$$C_\kappa^2(\mathcal{P}_0, \vec{\mathcal{P}}) - M^2 = 0, \qquad C_\kappa^2(\mathcal{Q}_0, \vec{\mathcal{Q}}) - M^2 = 0, \tag{70}$$

i.e. one should put the following on-shell energy values

$$\mathcal{P}_0^{(\pm)} = \pm\omega_\kappa(\vec{\mathcal{P}}), \qquad \mathcal{Q}_0^{(\pm)} = \pm\omega_\kappa(\vec{\mathcal{Q}}). \tag{71}$$

One can check that from (66) and (68) we get the Abelian addition laws for 3-momenta and energy

$$\vec{\mathcal{P}}^{(1+2)} = \vec{\mathcal{P}} + \vec{\mathcal{Q}} = \vec{\mathcal{P}}^{(2+1)}, \qquad \mathcal{P}_0^{(1+2)} = \mathcal{P}_0 + \mathcal{Q}_0 = \mathcal{P}_0^{(2+1)}, \tag{72}$$

where $\mathcal{P}_0, \mathcal{Q}_0$ lie on the mass shells (70). We point out here that in [1] we have chosen the Abelian addition law (72) as the selection principle for the choice of κ-deformed statistics.

It can be shown that for the choices given by (69) the κ-deformed oscillator algebra (61) can be written in the following standard classical form ($[A, B]_\circ := A \circ B - B \circ A$)[10]

$$[a_\kappa(\mathcal{P}), a_\kappa(\mathcal{Q})]_\circ = [a_\kappa^\dagger(\mathcal{P}), a_\kappa^\dagger(\mathcal{Q})]_\circ = 0, \quad [a_\kappa^\dagger(\mathcal{P}), a_\kappa(\mathcal{Q})]_\circ = 2\Omega_\kappa(\vec{\mathcal{P}})\delta^{(3)}(\vec{\mathcal{P}} - \vec{\mathcal{Q}}), \tag{73}$$

where the creation and annihilation operators occurring in (73) are ($\mathcal{P}_0^{(\pm)} = \pm\omega_\kappa(\vec{\mathcal{P}}); \mathcal{P}_0^+ > 0$)

$$a_\kappa(\mathcal{P}) := \hat{A}(\mathcal{P}_0^{(+)}, \vec{\mathcal{P}}), \qquad a_\kappa^\dagger(\mathcal{P}) := \hat{A}(\mathcal{P}_0^{(-)}, \vec{\mathcal{P}}) = \hat{A}(-\mathcal{P}_0^{(+)}, -\vec{\mathcal{P}});$$

$$\Omega_\kappa(\vec{\mathcal{P}}) \equiv \kappa \sinh\left(\frac{\omega_\kappa(\vec{\mathcal{P}})}{\kappa}\right) \tag{74}$$

and the κ-deformed $\circ$-multiplication of two oscillators was given in [1, 2].

The κ-deformed multiplication $\circ$ is defined in such a way that the following relation is valid [25] :

$$\hat{\varphi}(x)\hat{\star}_\kappa\hat{\varphi}(y) = \hat{\varphi}(x) \circ \hat{\varphi}(y), \tag{75}$$

where we define

$$\hat{\varphi}(x) \circ \hat{\varphi}(y) = \frac{1}{(2\pi)^3} \int \mathrm{d}^4p \, \mathrm{d}^4q \, \mathrm{e}^{\mathrm{i}(p_\mu x^\mu + q_\mu y^\mu)} \hat{A}(p_0, \vec{p}) \circ \hat{A}(q_0, \vec{q}) \cdot$$
$$\cdot \delta(C_\kappa^2(p_0, \vec{p}) - M^2)\delta(C^2(q_0, \vec{q}) - M^2). \tag{76}$$

Subsequently

$$[\hat{\varphi}(x), \hat{\varphi}(x)]_{\star_\kappa} = [\hat{\varphi}(x), \hat{\varphi}(y)]_\circ, \tag{77}$$

---

[10] In fact the rhs of second relation (73) is multiplied by the Planck constant $\hbar$. In our consideration we put $\hbar = 1$.

which after insertion of the relations (73) permits us to obtain the c-number value of the field commutator described by the formulae (21) and (22).

One can point out that in consistency with the classical 4-momenta addition law (72) the choice (69) is the one which provides the particular c-number field commutator function (22) which is invariant under the classical four-dimensional translations $x_\mu \rightarrow x_\mu + a_\mu$, i.e. we obtain the κ-deformed commutator function depending on the four-coordinate difference $x_\mu - y_\mu$ [12].

3.2. The Non-Abelian κ-deformed addition law and the κ-deformed flip operator

We choose now the arbitrary functions in (61) as follows:

$$f = 1, \quad g = 0, \quad k = 0, \quad l = 1; \quad p_0 = \mathcal{P}_0, \quad q_0 = \mathcal{Q}_0. \tag{78}$$

The mass-shell conditions (60) take the form

$$C_\kappa^2(\mathcal{P}_0, \vec{\mathcal{P}}e^{\mathcal{Q}_0/2\kappa}) - M^2 = 0, \qquad C_\kappa^2(\mathcal{Q}_0, \vec{\mathcal{Q}}e^{-\mathcal{P}_0/2\kappa}) - M^2 = 0, \tag{79}$$

or more explicitly

$$\mathcal{P}_0 = \epsilon\omega_\kappa(\vec{\mathcal{P}}e^{\mathcal{Q}_0/2\kappa}), \qquad \mathcal{Q}_0 = \epsilon'\omega_\kappa(\vec{\mathcal{Q}}e^{-\mathcal{P}_0/2\kappa}). \tag{80}$$

Then one obtains from (61) the following uniquely determined κ-deformed oscillator algebra

$$\hat{A}(\mathcal{P}_0, \vec{\mathcal{P}})\hat{A}(\mathcal{Q}_0, \vec{\mathcal{Q}}) - J\left(\begin{smallmatrix} p',q' \\ \mathcal{P},\mathcal{Q} \end{smallmatrix}\right)\hat{A}(\mathcal{Q}_0, \vec{\mathcal{Q}}e^{-\frac{\mathcal{P}_0}{\kappa}})\hat{A}(\mathcal{P}_0, \vec{\mathcal{P}}e^{\frac{\mathcal{Q}_0}{\kappa}}) = c - \text{number}, \tag{81}$$

where the c-number in (81) is proportional to the following Dirac delta

$$\delta^{(3)}(\vec{\mathcal{P}} - \vec{\mathcal{Q}}) \rightarrow \delta^{(3)}(\vec{\mathcal{P}}e^{\mathcal{Q}_0/2\kappa} - \vec{\mathcal{Q}}e^{-\mathcal{P}_0/2\kappa}), \tag{82}$$

and occurs only in the creation–annihilation (or annihilation–creation) sector (see (A.6) and (A.7) in the appendix). Formula (81) can be written as well with the use of the κ-deformed flip operator

$$\hat{A}(\mathcal{P}_0, \vec{\mathcal{P}})\hat{A}(\mathcal{Q}_0, \vec{\mathcal{Q}}) - \hat{\tau}_\kappa\big(\hat{A}(\mathcal{P}_0, \vec{\mathcal{P}})\hat{A}(\mathcal{Q}_0, \vec{\mathcal{Q}})\big) = c - \text{number}, \tag{83}$$

where

$$\hat{\tau}_\kappa\big(\hat{A}(\mathcal{P}_0, \vec{\mathcal{P}})\hat{A}(\mathcal{Q}_0, \vec{\mathcal{Q}})\big) = J\left(\begin{smallmatrix} p',q' \\ \mathcal{P},\mathcal{Q} \end{smallmatrix}\right)\hat{A}(\mathcal{Q}_0, \vec{\mathcal{Q}}e^{-\frac{\mathcal{P}_0}{\kappa}})\hat{A}(\mathcal{P}_0, \vec{\mathcal{P}}e^{\frac{\mathcal{Q}_0}{\kappa}}), \tag{84}$$

and $\mathcal{P}_0, \mathcal{Q}_0$ are the solutions of equations (80). Using twice the formula (84) and the property $J\left(\begin{smallmatrix} p,q \\ \mathcal{P},\mathcal{Q} \end{smallmatrix}\right)J\left(\begin{smallmatrix} \mathcal{P},\mathcal{Q} \\ p,q \end{smallmatrix}\right) = 1$ it is easy to see that the flip operator $\hat{\tau}_\kappa$ satisfies the condition

$$\hat{\tau}_\kappa^2 = 1. \tag{85}$$

In order to adjust the set of relations (A.4)–(A.7) (see the appendix) to the choice (78) of the functions f, g, k, l, p_0 and q_0 we should solve equations (80). The equation for the solutions $\mathcal{P}_0^{(\epsilon,\epsilon')} = \mathcal{P}_0^{(\epsilon,\epsilon')}(\vec{\mathcal{P}}, \vec{\mathcal{Q}})$ looks as follows:

$$\mathcal{P}_0^{(\epsilon,\epsilon')} = \epsilon\omega_\kappa\left(\vec{\mathcal{P}}\,\exp\left[\frac{\epsilon'}{2\kappa}\omega_\kappa\left(\vec{\mathcal{Q}}\,\exp\left(-\mathcal{P}_0^{(\epsilon,\epsilon')}/2\kappa\right)\right)\right]\right). \tag{86}$$

One obtains the second set of energy values using the relation

$$\mathcal{Q}_0^{(\epsilon,\epsilon')}(\vec{\mathcal{Q}}, \vec{\mathcal{P}}; \kappa) = \mathcal{P}_0^{(\epsilon,\epsilon')}(\vec{\mathcal{P}}, \vec{\mathcal{Q}}; -\kappa). \tag{87}$$

The formulae for $\mathcal{P}_0^{(\epsilon,\epsilon')}$ and $\mathcal{Q}_0^{(\epsilon,\epsilon')}$ after inserting in (81) provide an explicit example of a four set (A.4)–(A.7) (see the appendix) of the κ-deformed twisted oscillator algebra for creation and annihilation operators. After inserting these algebraic relations in the field commutator (59) we can show that the commutator (57) one obtains a c-number value.

4. A general algebraic structure of κ-deformed oscillator algebras

Firstly, let us consider standard undeformed theory. The standard algebra describing the bosonic oscillators $a(\mathcal{P})$, $a^\dagger(\mathcal{Q})$ looks as follows:

$$[a^\dagger(\mathcal{P}), a^\dagger(\mathcal{Q})] = [a^\dagger(\mathcal{P}), a^\dagger(\mathcal{Q})] = 0, \quad [a^\dagger(\mathcal{P}), a^\dagger(\mathcal{Q})] = 2\omega(\vec{\mathcal{P}})\delta^3(\vec{\mathcal{P}} - \vec{\mathcal{Q}}), \tag{88}$$

where $\mathcal{P} \equiv (\mathcal{P}_0 = \omega(\vec{\mathcal{P}}), \vec{\mathcal{P}})$.

The κ-deformation of the algebra (88) can be introduced in two ways:

(i) The deformation of oscillator algebra (88) described by the general κ-deformed multiplication rule.

In order to describe the general algebra (61) we introduce new κ-deformed multiplication $\odot$ generalizing the $\circ$-multiplication given in [1, 2] as follows:

$$\hat{A}(\mathcal{P})|_{\mathcal{P}_0=\epsilon\omega_\kappa(\vec{\mathcal{P}})} \odot \hat{A}(\mathcal{Q})|_{\mathcal{P}_0=\epsilon'\omega_\kappa(\vec{\mathcal{P}})} := J\begin{pmatrix} p(\mathcal{P},\mathcal{Q}), q(\mathcal{P},\mathcal{Q}) \\ \mathcal{P}, \mathcal{Q} \end{pmatrix} \tag{89}$$

$$\cdot \hat{A}(p_0^{(\epsilon,\epsilon')}(\mathcal{P}, \mathcal{Q}), \vec{p}(\mathcal{P}, \mathcal{Q}))\hat{A}(q_0^{(\epsilon,\epsilon')}(\mathcal{P}, \mathcal{Q}), \vec{q}(\mathcal{P}, \mathcal{Q}))|_{\mathcal{P}_0=\tilde{\pi}, \mathcal{Q}_0=\tilde{\rho}},$$

where $\epsilon = \epsilon' = 1$ describes the creation–creation sector, $\epsilon = -\epsilon'$ the creation–annihilation sector and $\epsilon = \epsilon' = -1$ the annihilation–annihilation sector.

We described in such a way the first part of binary relation (61) which depends on six arbitrary functions. One can consider a subclass of the multiplication rule (89) which permits to describe the relation (61) as the standard oscillator algebra (88) with the generalized κ-deformed multiplication. For such a purpose the multiplication (89) should satisfy an additional relation permitting as well to express the second part of the binary relation (61) by the use of the multiplication rule (89), namely

$$\hat{A}(\mathcal{Q})|_{\mathcal{Q}_0=\epsilon'\omega_\kappa(\vec{\mathcal{Q}})} \odot \hat{A}(\mathcal{P})|_{\mathcal{P}_0=\epsilon\omega_\kappa(\vec{\mathcal{P}})} = J\begin{pmatrix} p'(\mathcal{P},\mathcal{Q}), q'(\mathcal{P},\mathcal{Q}) \\ \mathcal{P}, \mathcal{Q} \end{pmatrix}$$

$$\cdot \hat{A}\left(q_0^{(\epsilon,\epsilon')}(\mathcal{P}, \mathcal{Q}), \vec{q}(\mathcal{P}, \mathcal{Q})e^{-p_0^{(\epsilon,\epsilon')}(\mathcal{P},\mathcal{Q})/\kappa}\right)$$

$$\cdot \hat{A}\left(p_0^{(\epsilon,\epsilon')}(\mathcal{P}, \mathcal{Q}), \vec{p}(\mathcal{P}, \mathcal{Q})e^{q_0^{(\epsilon,\epsilon')}(\mathcal{P},\mathcal{Q})/\kappa}\right)|_{\mathcal{P}_0=\tilde{\pi}, \mathcal{Q}_0=\tilde{\rho}}. \tag{90}$$

The validity of (90) restricts six arbitrary functions occurring in (61) in the following way:

$$p_i^T(\mathcal{P}, \mathcal{Q}) \equiv p_i(\mathcal{Q}, \mathcal{P}) = q_i(\mathcal{P}, \mathcal{Q})e^{-p_0^{(\epsilon,\epsilon')}(\mathcal{P},\mathcal{Q})/\kappa},$$
$$\left(p_0^{(\epsilon,\epsilon')}\right)^T(\mathcal{P}, \mathcal{Q}) = q_0^{(\epsilon,\epsilon')}(\mathcal{P}, \mathcal{Q}), \tag{91}$$

$$q_i^T(\mathcal{P}, \mathcal{Q}) \equiv q_i(\mathcal{Q}, \mathcal{P}) = p_i(\mathcal{P}, \mathcal{Q})e^{q_0^{(\epsilon,\epsilon')}(\mathcal{P},\mathcal{Q})/\kappa},$$
$$\left(q_0^{(\epsilon,\epsilon')}\right)^T(\mathcal{P}, \mathcal{Q}) = p_0^{(\epsilon,\epsilon')}(\mathcal{P}, \mathcal{Q}). \tag{92}$$

The algebra (61) with the choice (91) and (92) can be written as follows:

$$[a(\mathcal{P}), a(\mathcal{Q})]_\odot = [a^\dagger(\mathcal{P}), a^\dagger(\mathcal{Q})]_\odot = 0, \qquad [a^\dagger(\mathcal{P}), a(\mathcal{Q})]_\odot = c-\text{number}. \tag{93}$$

The relations (93) describe the class of κ-deformed oscillator algebras with generalized κ-deformed multiplication, parametrized by three arbitrary functions solving the conditions (91) and (92).

In particular if $p_0 = \mathcal{P}_0$, $q_0 = \mathcal{Q}_0$ and we choose (see (69))

$$\vec{p}(\mathcal{P}, \mathcal{Q}) = e^{-\frac{\mathcal{Q}_0}{2\kappa}}\vec{\mathcal{P}}, \qquad \vec{q}(\mathcal{P}, \mathcal{Q}) = e^{\frac{\mathcal{P}_0}{2\kappa}}\vec{\mathcal{Q}}, \tag{94}$$

we obtain

$$\vec{p}^T(\mathcal{P}, \mathcal{Q}) = \mathrm{e}^{-\frac{\mathcal{P}_0}{2\kappa}} \vec{\mathcal{Q}}, \qquad \vec{q}^T(\mathcal{P}, \mathcal{Q}) = \mathrm{e}^{\frac{\mathcal{Q}_0}{2\kappa}} \vec{\mathcal{P}}, \tag{95}$$

where $\mathcal{P}_0 = \mathcal{P}_0^{(\epsilon)}$, $\mathcal{Q}_0 = \mathcal{Q}_0^{(\epsilon')}$.

We see that the choice (69) which provides the first example of κ-multiplication [1] obviously satisfies the conditions (91) and (92).

(ii) The general κ-deformed flip operator.

Let us describe the commutator of the two standard bosonic oscillators as follows:

$$[\hat{A}(\mathcal{P}), \hat{A}(\mathcal{Q})] \equiv \hat{A}(\mathcal{P})\hat{A}(\mathcal{Q}) - \hat{\tau}_0[\hat{A}(\mathcal{P})\hat{A}(\mathcal{Q})] = c - \text{number}, \tag{96}$$

where $\mathcal{P}_0 = \epsilon\omega(\vec{\mathcal{P}})$ and $\hat{\tau}_0(\hat{A}\hat{B}) = \hat{B}\hat{A}$. In particular if $n = 2$, the symmetrization operator S_n is described in terms of the flip operator as follows:

$$S_2^{(0)} = \tfrac{1}{2}(1 + \hat{\tau}_0). \tag{97}$$

One can look for the κ-deformation of the classical flip operator $\hat{\tau}_0$, which leads to the following κ-deformation of the commutator (96)

$$[\hat{A}, \hat{B}] \to [\hat{A}, \hat{B}]_{\hat{\tau}_\kappa} := \hat{A}\hat{B} - \hat{\tau}_\kappa \hat{A}\hat{B}, \tag{98}$$

where we assume that

$$[\hat{\tau}_\kappa, \Delta(P_\mu)] = 0. \tag{99}$$

We supplement the multiplication (89) with the following definition of the κ-deformed flip operation

$$\hat{\tau}_\kappa \left[\hat{A}(\mathcal{P})|_{\mathcal{P}_0 = \epsilon\omega_\kappa(\vec{\mathcal{P}})} \odot \hat{A}(\mathcal{Q})|_{\mathcal{Q}_0 = \epsilon'\omega_\kappa(\vec{\mathcal{Q}})} \right]$$

$$\equiv \hat{\tau}_\kappa \left[J \begin{pmatrix} p(\mathcal{P},\mathcal{Q}), q(\mathcal{P},\mathcal{Q}) \\ \mathcal{P}, \mathcal{Q} \end{pmatrix} \hat{A}\big(p_0^{(\epsilon,\epsilon')}(\mathcal{P}, \mathcal{Q}), \vec{p}(\mathcal{P}, \mathcal{Q})\big)\hat{A}\big(q_0^{(\epsilon,\epsilon')}(\mathcal{P}, \mathcal{Q}), \vec{q}(\mathcal{P}, \mathcal{Q})\big) \right]|_{\mathcal{P}_0 = \tilde{\pi}, \mathcal{Q}_0 = \tilde{\rho}}$$

$$:= J \begin{pmatrix} p'(\mathcal{P},\mathcal{Q}), q'(\mathcal{P},\mathcal{Q}) \\ \mathcal{P}, \mathcal{Q} \end{pmatrix} \hat{A}\big(q_0^{(\epsilon,\epsilon')}(\mathcal{P}, \mathcal{Q}), \vec{q}(\mathcal{P}, \mathcal{Q})\mathrm{e}^{-p_0^{(\epsilon,\epsilon')}(\mathcal{P},\mathcal{Q})/\kappa}\big)$$

$$\cdot \hat{A}\big(p_0^{(\epsilon,\epsilon')}(\mathcal{P}, \mathcal{Q}), \vec{p}(\mathcal{P}, \mathcal{Q})\mathrm{e}^{q_0^{(\epsilon,\epsilon')}(\mathcal{P},\mathcal{Q})/\kappa}\big)|_{\mathcal{P}_0 = \tilde{\pi}, \mathcal{Q}_0 = \tilde{\rho}}, \tag{100}$$

consistently with (99). Further, one can show that for any choice of the multiplication $\odot$ we obtain

$$\hat{\tau}_\kappa^2 = 1. \tag{101}$$

In particular one can consider a κ-statistics described only by the κ-deformed flip operator with the standard undeformed multiplication rule

$$\hat{A}(\mathcal{P}) \odot \hat{A}(\mathcal{Q}) \equiv \hat{A}(\mathcal{P})\hat{A}(\mathcal{Q}). \tag{102}$$

Such a case is obtained by putting $f = l = 1$, $g = k = 0$ and $p_0 = \mathcal{P}_0$, $q_0 = \mathcal{Q}_0$, and the flip operator is uniquely defined as follows (see (84)):

$$\hat{\tau}_\kappa[\hat{A}(\mathcal{P})\hat{A}(\mathcal{Q})] = J \begin{pmatrix} p',q' \\ \mathcal{P}, \mathcal{Q} \end{pmatrix} \hat{A}\big(\mathcal{Q}_0, \vec{\mathcal{Q}}\mathrm{e}^{-\frac{\mathcal{P}_0}{\kappa}}\big)\hat{A}\big(\mathcal{P}_0, \vec{\mathcal{P}}\mathrm{e}^{\frac{\mathcal{Q}_0}{\kappa}}\big). \tag{103}$$

(iii) The most general κ-deformed oscillator algebra

We shall consider now the most general κ-deformed oscillator algebra (61), which for the energy–momentum dispersion relation described by the mass-shell conditions (60) leads to the c-number commutator function. It appears that such an algebra can be described by the composition of the multiplication (89) and the κ-deformed flip (100) in the following way:

$$(1 - \hat{\tau}_\kappa) \left[\hat{A}(\mathcal{P})|_{\mathcal{P}_0 = \epsilon\omega_\kappa(\vec{\mathcal{P}})} \odot \hat{A}(\mathcal{Q})|_{\mathcal{Q}_0 = \epsilon'\omega_\kappa(\vec{\mathcal{Q}})} \right] = c - \text{number}. \tag{104}$$

If we describe the 2-particle sector of the standard bosonic Hilbert space $\mathcal{H}_2$ as a symmetrized tensor product of 1-particle Hilbert spaces $\mathcal{H}_1$,

$$\mathcal{H}_2 = S_2 \circ (\mathcal{H}_1 \otimes \mathcal{H}_1), \tag{105}$$

the κ-deformed multiplication modifies the tensor product as follows:

$$\mathcal{H}_1 \otimes \mathcal{H}_1 \rightarrow \mathcal{H}_1 \otimes_\kappa \mathcal{H}_1, \tag{106}$$

where $\otimes_\kappa$ is obtained from the 'braided multiplication' $\odot$ of oscillators (see (89)) and κ-deformed twist $\hat{\tau}_\kappa$ changes the symmetrization operator

$$S_2 \rightarrow S_2^\kappa = \tfrac{1}{2}(1 + \hat{\tau}_\kappa). \tag{107}$$

In this paper we limited our considerations to the most general binary algebraic relations. Only in the case of algebra (61) with the particular choice of arbitrary functions given by (69) (see [1, 2]) the products of arbitrary numbers of κ-deformed creation and annihilation operators have been introduced, and all sectors of the corresponding κ-deformed Hilbert space have been considered.

5. Final remarks

In this paper we consider the general structure of κ-deformed binary oscillator algebras and their applicability for the description of free κ-deformed quantum fields, with c-number commutator functions. Our considerations generalize the recent examples of the κ-deformed statistics studied by the present authors [1, 2, 25], Arzano and Marciano [3], Young and Zegers [4, 5] and Govindarajan *et al* [16]. We consider two different classes of the κ-deformed oscillators which differ by their κ-deformed energy–momentum relations: in section 2 we assume that the oscillators $\hat{A}(p_0, \vec{p})$ are put on the standard κ-deformed mass shell (4), and in section 3 we modify the standard κ-deformed mass-shell conditions in a way providing c-number field commutators.

The basic results obtained in this paper are the following:

(i) We considered in section 2 the algebra of deformed oscillators on the κ-deformed mass-shell, corresponding to noncommutative field theory with standard multiplication of noncommutative fields, via standard κ-star product (10). In such a case for arbitrary choice of the κ-deformed oscillator algebra, the κ-deformed quantum fields are not free. One can show that the commutator function is necessarily bilinear in field operators.

(ii) We studied in section 3 the general algebra of oscillators with modified κ-deformed mass-shell conditions (see (79)), corresponding to noncommutative field theory with a modified multiplication law of noncommutative fields (see (14)) represented by nonstandard κ-star product (15). In such a case one obtains for large class of κ-deformed oscillator algebras (see (61) and Appendix), the κ-deformed free quantum fields which are characterized by the c-number commutator function. It should be stressed that only in such a case can one look for the formulation of the κ-deformed perturbative description of interacting κ-deformed field theory and consider the suitable generalization of the Feynman diagram technique.

(iii) In section 4 we did show that the general κ-deformed statistics providing κ-deformed quantum fields with c-number commutators is obtained by the composition of general κ-deformed multiplication and the κ-deformed flip operator.

We point out finally that in this paper we do not discuss the κ-covariance of the κ-deformed oscillator algebras. The results of [4, 5] suggest however that the κ-covariance is not consistent with the relation (43) which we had to postulate in order to obtain the c-number κ-fields commutator. It appears therefore that the freedom of choice present in the formula (61) might not permit to obtain both the manifest κ-covariance and the c-number commutator function.

Acknowledgments

Appendix A. general algebra of κ-deformed creation and annihilation operators

One can rewrite the mass-shell conditions (60) as the set of four nonlinear algebraic equations describing eight classes of energy–momentum dispersion relations $p_0 \equiv p_0^{(\epsilon,\epsilon')}(\tilde{\pi}, \vec{\mathcal{P}}; \tilde{\rho}, \vec{\mathcal{Q}})$, $q_0 \equiv q_0^{(\epsilon,\epsilon')}(\tilde{\pi}, \vec{\mathcal{P}}; \tilde{\rho}, \vec{\mathcal{Q}})$ ($\epsilon = \pm 1, \epsilon' = \pm 1$)

$$p_0 = \epsilon \omega_\kappa \big([\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g]e^{q_0/2\kappa}\big), \qquad q_0 = \epsilon' \omega_\kappa \big([\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l]e^{-p_0/2\kappa}\big), \qquad \text{(A.1)}$$

where

$$\lim_{\kappa \to \infty} p_0^{(\epsilon,\epsilon')} = \epsilon \omega(\vec{p}), \qquad \lim_{\kappa \to \infty} q_0^{(\epsilon,\epsilon')} = \epsilon' \omega(\vec{q}). \qquad \text{(A.2)}$$

The values $p_0^{(1,\epsilon')} > 0$, $\big(q_0^{(\epsilon,1)} > 0\big)$ describe the on-shell values of $p_0(q_0)$ which select $\hat{A}\big(p_0^{(1,\epsilon')}, \vec{p}\big)$, $\hat{A}\big(q_0^{(\epsilon,1)}, \vec{q}\big)$ as κ-deformed creation operators; the on-shell energy values $p_0^{(-1,\epsilon')} < 0$, $\big(q_0^{(\epsilon,-1)} < 0\big)$ are required to define the κ-deformed annihilation operators $\hat{A}\big(p_0^{(-1,\epsilon')}, \vec{p}\big)$, $\hat{A}\big(q_0^{(\epsilon,-1)}, \vec{q}\big)$. If we denote

$$p^{(\epsilon,\epsilon')} = \big(p_0^{(\epsilon,\epsilon')}, \vec{p}(\tilde{\pi}, \vec{\mathcal{P}}; \tilde{\rho}, \vec{\mathcal{Q}})\big), \qquad q^{(\epsilon,\epsilon')} = \big(q_0^{(\epsilon,\epsilon')}, \vec{q}(\tilde{\pi}, \vec{\mathcal{P}}; \tilde{\rho}, \vec{\mathcal{Q}})\big), \qquad \text{(A.3)}$$

one can rewrite the relations (61) as the four set of relations describing creation–creation, creation–annihilation, annihilation–creation and annihilation–annihilation sectors.

   If we insert in proper way the on-shell energy values $p_0^{(\epsilon,\epsilon')}$, $q_0^{(\epsilon,\epsilon')}$ (see (A.1) and (A.3)) the relations (61) decompose into the following set of relations:

 (i) the creation–creation algebra

$$J\binom{p^{(++)},q^{(++)}}{\mathcal{P},\mathcal{Q}} \hat{A}(p_0^{(++)}, \vec{\mathcal{P}}f + \vec{\mathcal{Q}}g) \cdot \hat{A}(q_0^{(++)}\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l)$$

$$= J\binom{p'^{(++)},q'^{(++)}}{\mathcal{P},\mathcal{Q}} \hat{A}\big(q_0^{(++)}, [\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l]e^{-p_0^{(++)}/\kappa}\big) \cdot \hat{A}\big(p_0^{(++)}, [\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g]e^{q_0^{(++)}/\kappa}\big),$$

$$\text{(A.4)}$$

 (ii) the annihilation–annihilation algebra

$$J\left(^{p^{(--)},q^{(--)}}_{\mathcal{P},\mathcal{Q}}\right) \hat{A}\big(p_0^{(--)}, \vec{\mathcal{P}}f + \vec{\mathcal{Q}}g\big) \cdot \hat{A}\big(q_0^{(--)}\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l\big)$$

$$= J\left(^{p'^{(--)},q'^{(--)}}_{\mathcal{P},\mathcal{Q}}\right) \hat{A}\big(q_0^{(--)}, [\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l]e^{-p_0^{(--)}/\kappa}\big) \cdot \hat{A}\big(p_0^{(--)}, [\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g]e^{q_0^{(--)}/\kappa}\big),$$

$$\text{(A.5)}$$

(iii) the creation–annihilation algebra

$$
J\begin{pmatrix} p^{(+-)}, q^{(-+)} \\ \mathcal{P}, \mathcal{Q} \end{pmatrix} \hat{A}\big(p_0^{(+-)}, \vec{\mathcal{P}}f + \vec{\mathcal{Q}}g\big) \cdot \hat{A}\big(q_0^{(-+)}\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l\big)
$$

$$
-J\begin{pmatrix} p'^{(+-)}, q'^{(-+)} \\ \mathcal{P}, \mathcal{Q} \end{pmatrix} \hat{A}\big(q_0^{(-+)}, [\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l]\mathrm{e}^{-p_0^{(+-)}/\kappa}\big) \cdot \hat{A}\big(p_0^{(+-)}, [\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g]\mathrm{e}^{q_0^{(-+)}/\kappa}\big)
$$

$$
= 2N_\kappa^{(+-)}(\vec{\mathcal{P}}, \vec{\mathcal{Q}})\delta^{(3)}\big[(\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g)\mathrm{e}^{q_0^{(-+)}/\kappa} - (\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l)\mathrm{e}^{-p_0^{(+-)}/\kappa}\big], \tag{A.6}
$$

(iv) the annihilation–creation algebra

$$
J\begin{pmatrix} p^{(-+)}, q^{(+-)} \\ \mathcal{P}, \mathcal{Q} \end{pmatrix} \hat{A}\big(p_0^{(-+)}, \vec{\mathcal{P}}f + \vec{\mathcal{Q}}g\big) \cdot \hat{A}\big(q_0^{(+-)}\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l\big)
$$

$$
-J\begin{pmatrix} p'^{(-+)}, q'^{(+-)} \\ \mathcal{P}, \mathcal{Q} \end{pmatrix} \hat{A}\big(q_0^{(+-)}, [\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l]\mathrm{e}^{-p_0^{(-+)}/\kappa}\big) \cdot \hat{A}\big(p_0^{(-+)}, [\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g]\mathrm{e}^{q_0^{(+-)}/\kappa}\big)
$$

$$
= 2N_\kappa^{(-+)}(\vec{\mathcal{P}}, \vec{\mathcal{Q}})\delta^{(3)}\big[(\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g)\mathrm{e}^{q_0^{(+-)}/\kappa} - (\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l)\mathrm{e}^{-p_0^{(-+)}/\kappa}\big]. \tag{A.7}
$$

The 3-momentum Dirac delta in (A.6) and (A.7) describes in accordance with formula (66) the κ-deformed 3-momentum conservation law for the process of creation/annihilation of field quanta with the 3-momentum $(\vec{\mathcal{P}}f + \vec{\mathcal{Q}}g)$ and annihilation of the quanta with the momentum $(\vec{\mathcal{P}}k + \vec{\mathcal{Q}}l)$.

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